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Fermion quasi-spherical harmonics

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Abstract. Quasi-spherical harmonics, $Y_\ell^m(\theta, \phi)$ are derived and presented for half-odd-integer values of ℓ and m. The form of the ϕ factor is identical to that in the case of integer ℓ and m: $\exp(\mathrm{i}m\phi)$. However, the domain of these functions in the half-odd-integer case is $0 \leqslant \phi < 4\pi$ rather than the domain $0 \leqslant \phi < 2\pi$ in the case of integer ℓ and m (the true spherical harmonics). The form of the θ factor, $P_\ell^{|m|}(\theta)$ (an associated Legendre function) is (as in the integer case) the factor $(\sin\theta)^{|m|}$ multiplied by a polynomial in $\cos\theta$ of degree $(\ell - |m|)$ (an associated Legendre polynomial). A substantial difference between the spherical (integer ℓ and m) and quasi-spherical (half-odd-integer ℓ and m) Legendre functions is that the latter have an irrational factor of $\sqrt{\sin\theta}$ whereas the factor of the truly spherical functions is an integer power of $\sin\theta$. The domain of both the true and quasi-spherical associated Legendre functions is the same: $0 \leqslant \theta < \pi$. A table of the associated Legendre functions is presented for both integer and half-odd-integer values of ℓ and m, for $|m| = 0, \frac{1}{2}, 1 \ldots \frac{11}{2}$, and for $(\ell - |m|) = 0, 1, 2, 3, 4, 5$. The table displays the similarity between the functions for integer ℓ and m (which are well known) and those for half-odd-integer ℓ and m (which have not been recognized previously).

1. Introduction

The theory of angular momentum based upon the fundamental commutation relations [1, p 93; 2, p 309; 3, pp 107–12], produces the eigenfunctions of $\boldsymbol{L}^2$ (the square of the total angular momentum) and $\boldsymbol{L}_z$ (its z-component) with eigenvalues of $\hbar^2\ell(\ell+1)$ and $\hbar m$, respectively [1, section 5.4, pp 100–3]. This general theory leads (via the raising and lowering ladder operators) to the prediction of both integer and half-odd-integer values of the quantum numbers ℓ and m [1, section 5.4, pp 100–3; 2, p 311], with the manifold of eigenvalues defined by:

- $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$
- $m = -\ell, -(\ell-1), -(\ell-2), \ldots, +(\ell-2), +(\ell-1), +\ell$ for given ℓ.

The quantum numbers, ℓ and m, are half-odd-integers for the spin and total (spin + orbital) angular momentum of fermions, and are integers for fermion orbital angular momentum, and for boson spin and total angular momentum.

This general theory of angular momentum is *abstract* in the sense that the operators, $\boldsymbol{L}^2$ and $\boldsymbol{L}_z$, and their eigenfunctions, are not functions of any coordinates.

In the Schrödinger wave mechanics of the hydrogen atom the operators $\boldsymbol{L}^2$ and $\boldsymbol{L}_z$ (that represent the motion of the electron around the proton) are expressed in terms of spherical

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Table 1. Legendre functions $P_\ell^{|m|}(x)$ ($\ell = |m| + i$).

$\|m\|$	Factor	0	1	$i = 2$	$i = 3$	$i = 4$	$i = 5$
0	1	1	x	$1 - 3x^2$	$3x - 5x^3$	$3 - 30x^2 + 35x^4$	$15x - 70x^3 + 63x^5$
$\frac{1}{2}$	$(1-x^2)^{\frac{1}{4}}$	1	x	$1 - 4x^2$	$3x - 6x^3$	$3 - 36x^2 + 48x^4$	$15x - 80x^3 + 80x^5$
1	$(1-x^2)^{\frac{1}{2}}$	1	x	$1 - 5x^2$	$3x - 7x^3$	$3 - 42x^2 + 63x^4$	$15x - 90x^3 + 99x^5$
$\frac{3}{2}$	$(1-x^2)^{\frac{3}{4}}$	1	x	$1 - 6x^2$	$3x - 8x^3$	$3 - 48x^2 + 80x^4$	$15x - 100x^3 + 120x^5$
2	$(1-x^2)$	1	x	$1 - 7x^2$	$3x - 9x^3$	$3 - 54x^2 + 99x^4$	$15x - 110x^3 + 143x^5$
$\frac{5}{2}$	$(1-x^2)^{\frac{5}{4}}$	1	x	$1 - 8x^2$	$3x - 10x^3$	$3 - 60x^2 + 120x^4$	$15x - 120x^3 + 168x^5$
3	$(1-x^2)^{\frac{3}{2}}$	1	x	$1 - 9x^2$	$3x - 11x^3$	$3 - 66x^2 + 143x^4$	$15x - 130x^3 + 195x^5$
$\frac{7}{2}$	$(1-x^2)^{\frac{7}{4}}$	1	x	$1 - 10x^2$	$3x - 12x^3$	$3 - 72x^2 + 168x^4$	$15x - 140x^3 + 224x^5$
4	$(1-x^2)^2$	1	x	$1 - 11x^2$	$3x - 13x^3$	$3 - 78x^2 + 195x^4$	$15x - 150x^3 + 255x^5$
$\frac{9}{2}$	$(1-x^2)^{\frac{9}{4}}$	1	x	$1 - 12x^2$	$3x - 14x^3$	$3 - 84x^2 + 224x^4$	$15x - 160x^3 + 288x^5$
5	$(1-x^2)^{\frac{5}{2}}$	1	x	$1 - 13x^2$	$3x - 15x^3$	$3 - 90x^2 + 255x^4$	$15x - 170x^3 + 323x^5$
$\frac{11}{2}$	$(1-x^2)^{\frac{11}{4}}$	1	x	$1 - 14x^2$	$3x - 16x^3$	$3 - 96x^2 + 288x^4$	$15x - 180x^3 + 360x^5$

polar coordinates: r, θ, ϕ [4, p 207; 1, p 95], where they have the form:

$$\boldsymbol{L}^2 = -\hbar^2 \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right\} \qquad \boldsymbol{L}_z = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial\phi}. \tag{1}$$

The eigenfunctions in this coordinate representation are called spherical harmonics and denoted by $Y_\ell^m(\theta, \phi)$. This Schrödinger representation of angular momentum leads to the same manifold of eigenvalues as the general theory, except that ℓ and m are restricted to be integers [2, pp 313–15]. This restriction arises from the argument that $Y_\ell^m(\theta, \phi)$ should be a single-valued function of the coordinates [1, p 103].

The purpose of this paper is to derive and present the eigenfunctions of the coordinate-representation operators (1) corresponding to half-odd-integer values of ℓ and m. Their θ factors are compared with those of the well known eigenfunctions with integer ℓ and m in table 1.

Notwithstanding their algebraic similarity (apparent from table 1), the integer and half-integer functions have different domains: the integer functions are defined on a Euclidean sphere, whereas the half-integer functions are not. One way of of interpreting this difference is that the angle ϕ has a range of $0 \leqslant \phi < 4\pi$ for the half-integer functions compared with $0 \leqslant \phi < 2\pi$ for the integer functions. This difference in domain is related to the essential difference between orbital angular momentum and spin angular momentum. The interpretation of the half-odd-integer functions is discussed in section 3.2.

Archival presentations of the associated Legendre functions [5, p 332] involve generating functions and general formulae which in principle allow explicit expressions and numerical values to be obtained for any required values of ℓ and m. However, some of these formulae become undefined when ℓ and m are not integers, because they involve differentiation (with respect to θ) ℓ or m times; e.g. Rodrigues' formula [5, section 8.6.18, p 334].

Thus to facilitate a clear exposition, we derive these functions as solutions of the appropriate differential equations.

2. The θ and ϕ differential equation

The differential equation for the eigenfunctions, $Y(\theta, \phi)$ and eigenvalues, A, of the square of the total angular momentum, $\boldsymbol{L}^2$ is:

$$-\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}Y(\theta,\phi)=AY(\theta,\phi). \tag{2}$$

It is well known that the independent variables, θ and ϕ, are separable, and hence $Y(\theta, \phi)$ can be written as a product:

$$Y(\theta,\phi)=\Theta(\theta)\times\Phi(\phi). \tag{3}$$

This leads to the separated, ordinary differential equations:

$$\left\{\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)+A\sin^2\theta-B\right\}\Theta(\theta)=0 \tag{4}$$

and

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}+B\right\}\Phi(\phi)=0 \tag{5}$$

where B is the separation constant arising from the separation of θ from ϕ.

The solutions of (5) are obvious (and well known) to have the form:

$$\Phi(\phi)=\exp(\mathrm{i}m\phi) \tag{6}$$

in which m is a constant to be determined. Substitution of (6) into (5) yields:

$$(-m^2+B)\Phi(\phi)=0 \qquad \text{or since} \quad \Phi(\phi)\neq 0, \quad m^2=B. \tag{7}$$

This relationship between the exponent m and the separation constant B is necessary, but in itself it does not specify the solutions any further. It is noteworthy that B has *the same value* for the two different solutions of (5) having m values equal in magnitude but opposite in sign; e.g. for $m = +2$ and $m = -2$, $B = 4$.

In the case of orbital angular momentum one proceeds by noting that the angle ϕ takes any value within a complete circle ($0 \leqslant \phi \leqslant 2\pi$), and hence the appropriate condition is that the function $\Phi(\phi)$ shall have the same value when ϕ transits a complete circle [4, pp 208–9; 3, p 38]:

$$\Phi(\phi+2\pi)=\Phi(\phi) \qquad \exp(\mathrm{i}m[\phi+2\pi])=\exp(\mathrm{i}m\phi)\Rightarrow\exp(\mathrm{i}m2\pi)=1. \tag{8}$$

This requirement is necessary for $\Phi(\phi)$ to be a proper (i.e. single-valued) function of the points that comprise the surface of a sphere; it is met as long as m is any integer.

We relax this single-valuedness condition on $\Phi(\phi)$ by leaving m undefined at this stage of the analysis, except that m should be real in order for $\Phi(\phi)$ to be periodic (and hence non-singular). Relaxing the traditional single-valuedness condition on $\Phi(\phi)$, leads, however, to a different, non-classical interpretation of the angle ϕ (see section 3.2).

2.1. Solution of the θ equation

Having solved the ϕ differential equation and determined that the separation constant B has the value $B = m^2$, the θ differential equation (4) (after dividing by $\sin^2\theta$) becomes:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left[A-\frac{m^2}{\sin^2\theta}\right]\Theta=0. \tag{9}$$

Following the usual derivation, we transform (9) to a new independent variable x defined by:

$$x=\cos\theta\Rightarrow\sin\theta=\sqrt{1-x^2} \tag{10}$$

and rename the dependent variable $P(x)$; i.e. $\Theta(\theta) \equiv P(x)$. The range of θ, $0 \leqslant \theta \leqslant \pi$ becomes $+1 \geqslant x \geqslant -1$.

With this transformation the differential equation (9) becomes:

$$(1-x^2)\frac{\mathrm{d}^2 P}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}P}{\mathrm{d}x} + \left[A - \frac{m^2}{1-x^2}\right]P = 0. \tag{11}$$

The last term, $m^2/(1-x^2)$, must be removed in order to develop the solution as a polynomial in x. This is achieved (as is well known) by writing the solution $P(x)$ as the product of a known factor $(1-x^2)^\alpha$ and a to-be-determined factor, $P'(x)$, that is anticipated to be a polynomial in x:

$$P(x) = (1-x^2)^\alpha P'(x). \tag{12}$$

After substitution of the product form (12) and its derivatives, the differential equation for $P(x)$ (11) becomes an equivalent equation for the dependent variable $P'(x)$:

$$(1-x^2)\frac{\mathrm{d}^2 P'}{\mathrm{d}x^2} - (2+4\alpha)x\frac{\mathrm{d}P'}{\mathrm{d}x} + \left(A - 2\alpha + \left[\frac{4x^2\alpha^2 - m^2}{1-x^2}\right]\right)P' = 0. \tag{13}$$

The denominator $(1-x^2)$ will cancel out if we choose α as follows:

$$4\alpha^2 = m^2 \Rightarrow \alpha = \pm\frac{\sqrt{m^2}}{2} \Rightarrow \alpha = \pm\frac{|m|}{2} \tag{14}$$

and we choose the positive square-root:

$$\alpha = +\frac{|m|}{2} \tag{15}$$

because choosing $\alpha = -|m|/2$ would make the solution (12) infinite at the ends of the range: $x = \pm 1$.

With this choice for α, the differential equation (13) for $P'(x)$ becomes:

$$(1-x^2)\frac{\mathrm{d}^2 P'}{\mathrm{d}x^2} - 2(|m|+1)x\frac{\mathrm{d}P'}{\mathrm{d}x} + [A - |m|(|m|+1)]P' = 0. \tag{16}$$

We proceed to substitute a power series for $P'(x)$ as follows:

$$P'(x) = \sum_{i=0} a_i x^{i+k} \tag{17}$$

and equating the coefficient of *every power* of x to zero (since the powers of x are linearly independent of each other) produces a set of homogeneous linear equations that determine the initial index k, the coefficients a_i, and the eigenvalues of the separation constant A.

Bypassing some details [6], it turns out that choosing $k = 0$ produces all possible solutions, and thus we obtain the equation:

$$\sum_{i=0}(a_i[A - (|m|+i)((|m|+i+1)] + a_{i+2}[(i+2)(i+1)])x^i = 0. \tag{18}$$

Equating each power of x to zero produces a set of linear equations that determine the coefficients of the power series $\{a_i : i = 0, 1, 2 \ldots\}$ which are concisely written as the matrix equation:

$$\begin{pmatrix} T_{00} & 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & T_{11} & 0 & 6 & 0 & 0 & \cdots \\ 0 & 0 & T_{22} & 0 & 12 & 0 & \cdots \\ 0 & 0 & 0 & T_{33} & 0 & 20 & \cdots \\ 0 & 0 & 0 & 0 & T_{44} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & T_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{19}$$

which is summarized by the equation:

$$\boldsymbol{T} \cdot \boldsymbol{a} = \boldsymbol{0} \tag{20}$$

where the elements of the square matrix $\boldsymbol{T}$ are given by:

$$\begin{aligned} &T_{ii} = A - (|m| + i)(|m| + i + 1) \qquad T_{ii+2} = (i + 1)(i + 2) \\ &T_{ij} = 0 \qquad \text{if} \quad j \neq i \quad \text{and} \quad j \neq i + 2. \end{aligned} \tag{21}$$

2.2. *The eigenvalues of A*

Since the matrix $\boldsymbol{T}$ is triangular, its determinant is equal to the product of its diagonal elements, and hence the eigenvalues of A (denoted by A_i—defined as being those values of A that make the determinant of $\boldsymbol{T}$ zero) are obtained by equating any one of the diagonal elements to zero:

$$A_i = (|m| + i)(|m| + i + 1) \qquad \text{for} \quad i = 0, 1, 2, \ldots . \tag{22}$$

This expression for the eigenvalues of the separation constant A shows that for a given value of $|m|$ (for the two values of m: $m = \pm|m|$) there is a set of values of A that increase quadratically with increasing values of i:

$$A_i = (|m| + i)(|m| + i + 1) = i^2 + i(2|m| + 1) + |m|(|m| + 1) \qquad i = 0, 1, 2 \ldots . \tag{23}$$

We see that the smallest value of A_i is $|m|(|m|+1)$ and that there is no upper limit on i nor upon the value of A_i. The index i is the (mathematically) natural 'quantum number' for designating an eigenvalue A_i; the index i is the degree of the polynomial in x, $P'(x)$, in equation (17).

However, it has become customary in physics to designate an alternative quantum number, ℓ, defined by:

$$\begin{aligned} &\ell = |m| + i \Rightarrow A_i = (|m| + i)(|m| + i + 1) = \ell(\ell + 1) \\ &\ell = |m|, (|m| + 1), (|m| + 2) \ldots . \end{aligned} \tag{24}$$

Thus we see that ℓ can take any positive value beginning at $|m|$ for a given value of the quantum number m in $\exp(\mathrm{i}m\phi)$.

Furthermore it has also become customary in physics, to reverse the precedence of the relationship between ℓ and $|m|$ given in (24) by regarding ℓ as primary, with m as the secondary quantum number:

$$\ell = 0, 1, 2 \ldots \qquad A = \ell(\ell + 1) \qquad 0 \leqslant |m| \leqslant \ell \Rightarrow -\ell \leqslant m \leqslant +\ell. \tag{25}$$

This physical viewpoint arises from atomic spectroscopy wherein the energy of an atomic state depends upon ℓ, but all $2\ell + 1$ states differing only in m are degenerate in the absence of an external magnetic field [7, p 4].

For a given value of ℓ the minimum and maximum values of m correspond to $m_{min} = -\ell$ and $m_{max} = +\ell$. These $2\ell+1$ values of m correspond to polynomial degrees, i, differing by 1:

i	0	1	2	...	2	1	0
m	$-\ell$	$-\ell + 1$	$-\ell + 2$	...	$\ell - 2$	$\ell - 1$	ℓ

and hence these $2\ell+1$ values of m span an interval equal to 2ℓ that is symmetrical about zero. This interval is necessarily a non-negative integer, n, and hence:

$$2\ell = n \qquad \text{a non-negative integer} \tag{26}$$

and hence the allowed values of ℓ are:

$$\ell = \frac{n}{2} \tag{27}$$

where n is any non-negative integer. When n is even, ℓ is itself an integer, but when n is odd, ℓ is a half of an odd-integer. This is how both integer and half-odd-integer values of ℓ and m arise in the solution of the associated Legendre differential equation. This argument is essentially identical with that used in the abstract theory of angular momentum [2, p 311] to deduce that both integer, and half-odd-integer, values of ℓ and m are allowed.

2.3. The eigenfunctions $\{a_i : i = 0, 1, 2 \ldots\}$

Choosing $T_{ii} = 0$ in (21) (i.e. $A_i = [|m| + i][|m| + i + 1]$) will lead to a solution in which the only non-zero coefficients are a_i, and *all* lower coefficients of the same parity (i.e. odd or even). The ratios of these non-zero coefficients are given by the *recursion relation*:

$$[A_i - (|m| + k)(|m| + k + 1))]a_k + (k + 1)(k + 2)a_{k+2} = 0 \Rightarrow$$
$$a_k = -\frac{(k+1)(k+2)}{(i-k)(2|m| + i + k + 1)} a_{k+2} \qquad \text{for} \quad k = (i-2), (i-4) \ldots \{1 \text{ or } 0\}. \tag{28}$$

The recursion will terminate at $k = 1$ if i is odd, and at $k = 0$ if i is even. Since $k < i$ the factors in the denominator of (28) are always both positive, and hence consecutive terms of the power series in x alternate in sign. This recursion relation (28) is written as beginning with the highest-index coefficient a_i, from which the lower-index coefficients are calculated.

Alternatively, the recursion can be written as beginning with a_0 or a_1:

$$a_{k+2} = -\frac{(i-k)(2|m| + i + k + 1)}{(k+1)(k+2)} a_k \qquad \text{for} \quad k = 0 \quad \text{or} \quad 1 \ldots (i-2). \tag{29}$$

This recursion relation was used to generate table 1; the coefficient of x^0 or x^1 was chosen to be positive and all of the coefficients were re-normalized to make them the smallest set of integers for a given polynomial degree i.

Table 1 displays a striking similarity between the well known associated Legendre functions for integer values of ℓ and $|m|$, and the newly discovered functions for half-odd-integer values. One difference between them, is that the factor $(1 - x^2)^{\frac{|m|}{2}} = \sin^{|m|}\theta$ is an integer power of $\sin\theta$ when m is an integer, but that it has a factor of $\sqrt{\sin\theta}$ when m is a half-odd-integer; in the latter case the gradient, $\mathrm{d}P_\ell^{|m|}(\theta)/\mathrm{d}\theta$ is infinite at the limits: $\theta = 0$, $\theta = \pi$, whereas in the integer case these gradients are finite. The *polynomial*, $P_\ell^{\prime|m|}(\cos\theta)$, involves only integer powers of $x = \cos\theta$ in both cases.

2.4. Normalization of the spherical harmonics

The normalization constant, N^2, of $Y(\theta, \phi)$ (equation (3)) is defined by:

$$N^2 = \int_0^{2\pi} |\Phi(\phi)|^2 \,\mathrm{d}\phi \int_0^{\pi} [\Theta_\ell^{|m|}(\theta)]^2 \sin\theta \,\mathrm{d}\theta. \tag{30}$$

The ϕ factor of N^2 is always 2π because $|\Phi(\phi)|^2 = \exp(-\mathrm{i}m\phi)\exp(\mathrm{i}m\phi) = 1$. However, since the domain of the functions for the half-odd-integer values of m, is $0 \leqslant \phi < 4\pi$, the integration over ϕ involved in (30) should also, in principle, be over this range $0 \leqslant \phi < 4\pi$. This would only make the ϕ factor of N^2 equal to 4π rather than 2π, and so for the sake of consistency we choose to have the same range of integration for both the integral and half-integral values of m.

The θ factor of N^2 (N_θ^2) is given by:

$$N_\theta^2 = \int_0^{\pi} [\Theta_\ell^{|m|}(\theta)]^2 \sin\theta \,\mathrm{d}\theta = \int_{-1}^{+1} |P_\ell^{|m|}(x)|^2 \,\mathrm{d}x. \tag{31}$$

One difference between the case of integer values of ℓ, and that of half-odd-integer values, is that in the latter case N_θ^2 has a factor of π, whereas in the integer case it is a rational fraction. This is illustrated in table 2, whose values correspond to normalizing the functions shown in table 1.

Table 2. Normalization integrals N_θ^2 of Legendre functions $P_\ell^{|m|}(x)$.

	i					
$\|m\|$	0	1	2	3	4	5
0	$\frac{2}{1}$	$\frac{2}{3}$	$\frac{8}{5}$	$\frac{8}{7}$	$\frac{128}{9}$	$\frac{128}{11}$
$\frac{1}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{8}$	$\frac{\pi}{2}$	$\frac{9\pi}{32}$	$\frac{9\pi}{2}$	$\frac{25\pi}{8}$
1	$\frac{4}{3}$	$\frac{4}{15}$	$\frac{32}{21}$	$\frac{32}{45}$	$\frac{768}{55}$	$\frac{768}{91}$
$\frac{3}{2}$	$\frac{3\pi}{8}$	$\frac{\pi}{16}$	$\frac{15\pi}{32}$	$\frac{3\pi}{16}$	$\frac{35\pi}{8}$	$\frac{75\pi}{32}$
2	$\frac{16}{15}$	$\frac{16}{105}$	$\frac{64}{45}$	$\frac{192}{385}$	$\frac{6144}{455}$	$\frac{2048}{315}$
$\frac{5}{2}$	$\frac{5\pi}{16}$	$\frac{5\pi}{128}$	$\frac{7\pi}{16}$	$\frac{35\pi}{256}$	$\frac{135\pi}{32}$	$\frac{945\pi}{512}$
3	$\frac{32}{35}$	$\frac{32}{315}$	$\frac{512}{385}$	$\frac{512}{1365}$	$\frac{4096}{315}$	$\frac{20\,480}{3927}$
$\frac{7}{2}$	$\frac{35\pi}{128}$	$\frac{7\pi}{256}$	$\frac{105\pi}{256}$	$\frac{27\pi}{256}$	$\frac{2079\pi}{512}$	$\frac{385\pi}{256}$
4	$\frac{256}{315}$	$\frac{256}{3465}$	$\frac{1024}{819}$	$\frac{1024}{3465}$	$\frac{16\,384}{1309}$	$\frac{81\,920}{19\,019}$
$\frac{9}{2}$	$\frac{63\pi}{256}$	$\frac{21\pi}{1024}$	$\frac{99\pi}{256}$	$\frac{693\pi}{8192}$	$\frac{1001\pi}{256}$	$\frac{1287\pi}{1024}$
5	$\frac{512}{693}$	$\frac{512}{9009}$	$\frac{4096}{3465}$	$\frac{4096}{17\,017}$	$\frac{32\,768}{2717}$	$\frac{32\,768}{9009}$
$\frac{11}{2}$	$\frac{231\pi}{1024}$	$\frac{33\pi}{2048}$	$\frac{3003\pi}{8192}$	$\frac{143\pi}{2048}$	$\frac{3861\pi}{1024}$	$\frac{8775\pi}{8192}$

3. Discussion

The fermion quasi-spherical harmonics were discovered by Schlifer during a summer collaboration with Hunter several years ago [8]. More recent collaborative work involving the other authors of this paper led to the recognition of the potential utility of these functions for modelling the magnetic field of fermions such as the electron [6, 9].

The simplicity of the functions and their similarity to the well known spherical harmonics for integer ℓ and $|m|$ (table 1), suggested that they might have been discovered many years ago. However, a quite extensive literature search, including treatises on mathematical functions [5] and on theoretical physics [10], did not reveal any previous presentation of them.

They have existed in principle as special cases of the hypergeometric function [5, p 332 and pp 561–2], but:

(i) the polynomial nature of their factors, $P_\ell'^{|m|}(\cos\theta)$, and

(ii) their potential application as eigenfunctions of the spin angular momentum of fermion particles,

have not previously been recognized.

3.1. *Mathematical aspects*

It is beyond the scope of this paper to attempt a comprehensive consideration of these fermion spherical harmonics comparable with that in a standard treatise [5, pp 332–41]. Here we simply point out some of the salient mathematical aspects to be considered in adapting the general theory of Legendre functions to the fermion quasi-spherical harmonics.

(i) The formulae which are apparently undefined when ℓ and m are not integers (because they involve differentiation with respect to θ, ℓ or m times), notably Rodrigues' formula [5, section 8.6.18, p 334]:

$$P_\ell^0(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n (x^2 - 1)^n}{\mathrm{d}x^n}$$

and the formula involving differentiation w.r.t. m [5, section 8.6.6, p 334]:

$$P_\ell^{|m|}(x) = (-1)^{|m|}(1-x^2)^{\frac{|m|}{2}} \frac{\mathrm{d}^{|m|} P_\ell^0(x)}{\mathrm{d}x^{|m|}}$$

may be applicable by the theory of semi-differentiation [11, pp 115 and 307].

(ii) The recurrence relation [5, section 8.5, p 334]:

$$(\ell - |m| + 1) P_{\ell+1}^{|m|} = (2\ell + 1) x P_\ell^{|m|} - (\ell + |m|) P_{\ell-1}^{|m|}$$

is applicable to the case of half-odd-integer values of ℓ and $|m|$; it is noteworthy that all the coefficients in this recurrence relation: $(\ell - |m| + 1)$, $(2\ell + 1)$, $(\ell + |m|)$, are integers even when ℓ and m are half-odd-integers.

This recurrence relation produces a normalization and phase in which the *highest* power of x in each polynomial is *positive*, and in which the polynomial coefficients are generally fractions. A different normalization and phase was chosen to construct table 1 in order to display the regularity of the series of functions.

3.2. Interpretation

The mathematical way of defining the half-odd-integer, quasi-spherical harmonics to be proper (i.e. single-valued) functions of the angle ϕ, is to define its range to be $0 \leqslant \phi \leqslant 4\pi$, since:

$$\exp(\mathrm{i}m[\phi + 4\pi]) = +\exp(\mathrm{i}m\phi)$$

when m is half of an odd integer. This is concordant with the well known 4π symmetry of fermion wavefunctions; i.e the angle ϕ must transit two complete circles for the wavefunction to return to its original value [12, pp 21 and 138; 18, p 141].

However, with the range of ϕ redefined in this way, ϕ can no longer be regarded as one of the coordinate angles of the points on the surface of a sphere. Rather it is an angular coordinate of the points on a double-sphere, for which the points on the outer surface of the sphere are different from the corresponding points on the inner surface [16, p 419]. The geometry and topology of the double sphere can be modelled by the *Dirac belt trick* [16, p 417]. Such a closed-on-itself surface with 4π periodicity is the well known Möbius band [17, pp 141–3; 16, p 418]. Thus the fermion spherical harmonics are not true *spherical* harmonics, but rather *quasi-spherical* harmonics.

An alternative way of dealing with these functions is to define their domain as the spherical range $0 \leqslant \phi < 2\pi$, but then to recognize that they are double-valued functions of ϕ. This is the approach of Bethe's theory of double-groups as discussed by Altmann [13, chapter 13]. It has also been based upon the theory of non-simply connected spaces [14, section 3.2, pp 35–8].

In group-theoretical terms the integer ℓ and m functions are representations of $SO(3)$ while the half-integers functions are representations of $SU(2)$ [14, pp 35–41; 15, pp 123–30; 18, pp 140–2].

Regardless of whether one takes:

(i) the single-valued function on a non-spherical domain approach, or
(ii) the double-valued function on a spherical domain approach,

the angle ϕ does not have a classical, physical interpretation. Notwithstanding the long history of spin angular momentum [7], its physical nature remains something that is not easily interpreted in terms of rotational motion in classical space-time.

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References

[1] Rae A I M 1992 *Quantum Mechanics* 3rd edn (Bristol: IOP Publishing)
[2] Kaempffer F A 1965 *Concepts in Quantum Mechanics* (New York: Academic)
[3] Hannabuss K 1997 *An Introduction to Quantum Theory* (Oxford: Clarendon)
[4] McQuarrie D A 1983 *Quantum Chemistry* (Mill Valley, CA: University Science Books)
[5] Abramowitz M and Stegun I A 1964 *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Washington, DC: US Department of Commerce)
[6] Heck S 1994 *The Enigmatic Electron* (Toronto: York University) (undergraduate thesis)
[7] Tomonaga S I 1997 *The Story of Spin* translated by Takeshi Oka (Chicago, IL: University of Chicago Press)
[8] Schlifer I 1991 *Solution of Maxwell's Equations in Spherical Coordinates* (Toronto: York University) (internal report)
[9] Arslan S 1998 *Modelling the Electron as an Electromagnetic Wave* (Toronto: York University) (undergraduate thesis)
[10] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill)
[11] Miller K S and Ross B 1993 *An Introduction to the Fractional Calculus and Fractional Differential Equations* (New York: McGraw-Hill)
[12] Icke V 1995 *The Force of Symmetry* (Cambridge: Cambridge University Press)
[13] Altmann S L 1986 *Rotations, Quaternions, and Double Groups* (Oxford: Clarendon)
[14] Morandi G 1992 *The Role of Topology in Classical and Quantum Physics* (Berlin: Springer)
[15] Mackey G W 1963 *The Mathematical Foundations of Quantum Mechanics* (New York: Benjamin)
[16] Kauffman L H 1991 *Knots and Physics* (Singapore: World Scientific)
[17] Nash C and Sen S 1983 *Topology and Geometry for Physicists* (London: Academic)
[18] Jones H F 1990 *Groups, Representations and Physics* (Bristol: IOP Publishing)